Thurston norm and cosmetic surgeries

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Abstract

Two Dehn surgeries on a knot are called cosmetic if they yield homeomorphic manifolds. For a null-homologous knot with certain conditions on the Thurston norm of the ambient manifold, if the knot admits cosmetic surgeries, then the surgery coefficients are equal up to sign.

1 Introduction

Heegaard Floer homology is a powerful theory introduced by Ozsváth and Szabó [9]. One important aspect of Heegaard Floer homology is that it behaves well under Dehn surgeries. In fact, if one knows about the knot Floer complex of a knot, then one can compute the Heegaard Floer homology of any surgery on the knot [11, 14, 13]. This makes Heegaard Floer homology very useful in the study of Dehn surgery.

In this paper, we will use Heegaard Floer homology to study cosmetic surgeries. We first recall the definition of cosmetic surgeries.

Definition 1.1. If two Dehn surgeries on a knot yield homeomorphic manifolds, then these two surgeries are *cosmetic*.

Cosmetic surgeries are very rare. More precisely, one has the following Cosmetic Surgery Conjecture.

Conjecture 1.2. [5, Problem 1.81] *Suppose K is a knot in a closed manifold Y. If the complement of K is irreducible and is not the solid torus, then any two surgeries on K do not yield manifolds which are homeomorphic via an orientation preserving homeomorphism.*

The main theorem of this paper is an analogue of [13, Theorem 9.7] and [8, Theorem 1.5]. See also [16].

All manifolds in this paper are oriented, unless otherwise stated.

Theorem 1.3. *Suppose Y is a closed 3–manifold with $b_1(Y) > 0$. Let K be a null-homologous knot in Y, then the inclusion map $Y - K \to Y$ induces an isomorphism $H_2(Y - K) \cong H_2(Y)$, so we can identify $H_2(Y)$ with $H_2(Y - K)$.*

Suppose $r \in \mathbb{Q} \cup \{\infty\}$, let $Y_r(K)$ be the manifold obtained by r–surgery on K. Suppose (Y, K) satisfies that

$$x_Y(h) < x_{Y-K}(h), \quad \textit{for any nonzero element} \quad h \in H_2(Y). \tag{1}$$

Here x_M is the Thurston norm [15] in M. The conclusion is, if two rational numbers r, s satisfy that $Y_r(K) \cong \pm Y_s(K)$, then $r = \pm s$.

Sometimes the condition (1) can be weakened if there is a certain additional condition. For example, we can prove the following theorem.

Theorem 1.4. *Suppose Y is a closed 3–manifold with $b_1(Y) > 0$. Suppose K is a null-homologous knot in Y. Suppose $x_Y \equiv 0$, while the restriction of x_{Y-K} on $H_2(Y)$ is nonzero. Then we have the same conclusion as Theorem 1.3. Namely, if two rational numbers r, s satisfy that $Y_r(K) \cong \pm Y_s(K)$, then $r = \pm s$.*

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2 Non-triviality theorems

In this section, we will state some non-triviality theorems in Heegaard Floer homology. We first set up some notations we will use in this paper.

Let Y be a closed 3–manifold. Suppose $\mathfrak{S}$ is a subset of $\mathrm{Spin}^c(Y)$, let

$$HF^\circ(Y, \mathfrak{S}) = \bigoplus_{\mathfrak{s} \in \mathfrak{S}} HF^\circ(Y, \mathfrak{s}),$$

where HF° is one of $\widehat{HF}, HF^\infty, HF^+, HF^-$. Furthermore, if $h \in H_2(Y)$, then

$$HF^\circ(Y, h, i) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y), \langle c_1(\mathfrak{s}), h\rangle = 2i} HF^\circ(Y, \mathfrak{s}).$$

Similarly, if F is a Seifert surface for a knot $K \subset Y$, then

$$\widehat{HFK}(Y, K, [F], i) = \bigoplus_{\xi \in \underline{\mathrm{Spin}}^c(Y,K), \langle c_1(\xi), \widehat{F}\rangle = 2i} \widehat{HFK}(Y, K, \xi),$$

see [11] for more details. Following Kronheimer and Mrowka [7], let

$$HF^\circ(Y|h) = HF^\circ(Y, h, \frac{1}{2}x(h)).$$

A very important feature of Heegaard Floer homology is that it detects the Thurston norm of a 3–manifold. In [10], this result is stated for universally twisted Heegaard Floer homology. Nevertheless, this result should also hold if one uses untwisted coefficients. In fact, the analogous result for Monopole Floer homology is stated with untwisted coefficients [6, Corollary 41.4.2]. In order to state our results, we first recall two definitions.

Definition 2.1. Suppose M is a compact 3–manifold, a properly embedded surface $S \subset M$ is *taut* if $x(S) = x([S])$ in $H_2(M, \partial S)$, no proper subsurface of S is null-homologous, and if any component of S lies in a homology class that is represented by an embedded sphere then this component is a sphere. Here $x(\cdot)$ is the Thurston norm.

Definition 2.2. Suppose K is a null-homologous knot in a closed 3–manifold Y. An oriented surface $F \subset Y$ is a *Seifert-like surface* for K, if $\partial F = K$. When F is connected, we say that F is a *Seifert surface* for K. We also view a Seifert-like surface as a proper surface in $Y - \mathring{\nu}(K)$.

As in the proof of [2, Theorem 2.2], using the known non-triviality results for twisted coefficients stated in [8] and the Universal Coefficients Theorem, we can prove the following theorems. (The same results can also be proved via the approach taken in [4, 7].)

Theorem 2.3. *Suppose Y is a closed 3–manifold, $h \in H_2(Y)$, then*

$$HF^+(Y|h) \otimes \mathbb{Q} \neq 0, \quad \widehat{HF}(Y|h) \otimes \mathbb{Q} \neq 0.$$

Theorem 2.4. *Suppose K is a null-homologous knot in a closed 3–manifold Y. Let F be a taut Seifert-like surface for K. Then*

$$\widehat{HFK}(Y, K, [F], \frac{x(F)+1}{2}) \otimes \mathbb{Q} \neq 0.$$

3 A surgery formula

Suppose $K \subset Y$ is a null-homologous knot. Let $Y_{p/q}(K)$ denote the manifold obtained by $\frac{p}{q}$–surgery on K. Note that there is a natural identification

$$\mathrm{Spin}^c(Y_{p/q}(K)) \cong \mathrm{Spin}^c(Y) \times \mathbb{Z}/p\mathbb{Z}.$$

Let $\pi\colon\ \mathrm{Spin}^c(Y_{p/q}(K)) \to \mathrm{Spin}^c(Y)$ be the projection to the first factor.

The goal of this section is to prove the following theorem, which is a (much easier) analogue of [13, Theorem 1.1].

Theorem 3.1. *Suppose $K \subset Y$ is a null-homologous knot. If $\widehat{HF}(Y, \mathfrak{s}) = 0$, then there exists a constant $C = C(Y, K, \mathfrak{s})$, such that*

$$\mathrm{rank}\, \widehat{HF}(Y_{p/q}(K), \pi^{-1}(\mathfrak{s})) = qC.$$

3.1 Large surgeries on rationally null-homologous knots

Suppose $K \subset Y$ is a rationally null-homologous knot. We construct a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ for (Y, K), such that $\beta_1 = \mu$ is a meridian of K. Moreover, w, z are two base points associated with a marked point on β_1 as in [11]. There is a curve $\lambda \subset \Sigma$ which gives rise to the knot K. Doing oriented cut-and-pastes to λ and m parallel copies of μ, we get a connected simple closed curve supported in a small neighborhood of $\mu\cup\lambda$. We often denote this curve by $m\mu+\lambda$. The m parallel copies of μ are supported in a small neighborhood of μ. We call this neighborhood the *winding region* for $m\mu+\lambda$. $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma},z)$ is a diagram for $Y_{m\mu+\lambda}(K)$, where $\gamma_1=m\mu+\lambda$ and all other γ_i's are small Hamiltonian translations of β_i's.

Definition 3.2. As in [13, Section 4], one defines a map

$$\Xi\colon\ \mathrm{Spin}^c(Y_{m\mu+\lambda}(K))\to\underline{\mathrm{Spin}}^c(Y,K)$$

as follows. If $\mathfrak{t}\in\mathrm{Spin}^c(Y_{m\mu+\lambda}(K))$ is represented by a point $\mathbf{y}$ supported in the winding region, let $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ be the "nearest point", and let $\psi\in\pi_2(\mathbf{y},\Theta,\mathbf{x})$ be a small triangle. Then

$$\Xi(\mathfrak{t})=\underline{\mathfrak{s}}_{w,z}(\mathbf{x})+\big(n_w(\psi)-n_z(\psi)\big)\cdot\mu. \tag{2}$$

When we construct the Heegaard triple diagram

$$(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},w,z),$$

the position of the meridian β_1 relative to the points in $\lambda\cap\gamma_1$ may vary. Our next lemma says that the choice of the position of β_1 does not affect the definition of Ξ.

Lemma 3.3. *Suppose we have two Heegaard triple diagrams as above*

$$\Gamma_1=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^1,\boldsymbol{\gamma},w^1,z^1),\quad \Gamma_2=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}^2,\boldsymbol{\gamma},w^2,z^2).$$

The two sets $\boldsymbol{\beta}^1$ and $\boldsymbol{\beta}^2$ differ at the meridian, where the meridian $\beta_1^2\in\boldsymbol{\beta}^2$ is a parallel translation of the meridian $\beta_1^1\in\boldsymbol{\beta}^1$, still supported in the winding region. The two base points are moved together with the meridian.

Using these two diagrams, we can define two maps

$$\Xi^1,\Xi^2\colon\ \mathrm{Spin}^c(Y_{m\mu+\lambda}(K))\to\underline{\mathrm{Spin}}^c(Y,K).$$

Then $\Xi^1=\Xi^2$.

Proof. Without loss of generality, we may assume there is only one intersection point of $\lambda\cap\gamma_1$ between β_1^1 and β_1^2. See Figure 1 for an illustration.

Suppose $\mathbf{y}^1,\mathbf{y}^2\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma$ are two intersection points supported in the winding region, and suppose their γ_1–coordinates are y^1,y^2, respectively. Assume $\mathfrak{s}_{w^1}(\mathbf{y}^1)=\mathfrak{s}_{w^2}(\mathbf{y}^2)=\mathfrak{t}$, we want to prove that $\Xi^1(\mathfrak{t})=\Xi^2(\mathfrak{t})$.

By [9, Lemma 2.19],

$$\begin{aligned}\mathfrak{s}_{w^1}(\mathbf{y}^1)-\mathfrak{s}_{w^1}(\mathbf{y}^2)&=\mathrm{PD}(\varepsilon(\mathbf{y}^2,\mathbf{y}^1)),\\ \mathfrak{s}_{w^2}(\mathbf{y}^2)-\mathfrak{s}_{w^1}(\mathbf{y}^2)&=\mathrm{PD}(\mu).\end{aligned}$$

Hence $\varepsilon(\mathbf{y}^2,\mathbf{y}^1)=\mu$. Let $\tilde{\mathbf{y}}^1\in\mathbb{T}_\alpha\cap\mathbb{T}_\gamma$ be the point whose coordinates coincide with the coordinates of $\mathbf{y}^1$, except that its γ_1–coordinate is the next intersection

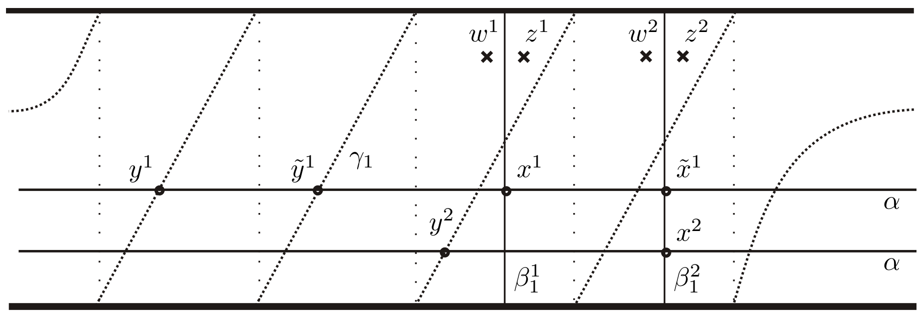

Figure 1: Local picture of the two triple Heegaard diagrams

point to y^1 on the same α–curve, denoted $\tilde{y}^1$. Then $\varepsilon(\tilde{\mathbf{y}}^1, \mathbf{y}^1) = \mu$, so $\tilde{\mathbf{y}}^1$ is in the same equivalence class as $\mathbf{y}^2$.

Now we only need to prove that

$$\Xi^1(\mathfrak{s}_{w^1}(\mathbf{y}^1)) = \Xi^2(\mathfrak{s}_{w^2}(\tilde{\mathbf{y}}^1)). \tag{3}$$

Let $\mathbf{x}^1 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^1}, \tilde{\mathbf{x}}^1 \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta^2}$ be the nearest points to $\mathbf{y}^1, \tilde{\mathbf{y}}^1$, respectively. It is clear that $\underline{\mathfrak{s}}_{w^1,z^1}(\mathbf{x}^1) = \underline{\mathfrak{s}}_{w^2,z^2}(\tilde{\mathbf{x}}^1)$. Moreover, the small triangle for $\tilde{\mathbf{y}}^1$ in Γ_2 is just a translation of the small triangle for $\mathbf{y}^1$ in Γ_1, so they contribute the same $n_w(\psi) - n_z(\psi)$ term in (2). So (3) follows. $\square$

Remark 3.4. In [13], in order to define $\Xi(\mathfrak{t})$, one places the meridian in a position such that the equivalence class of intersection points representing $\mathfrak{t}$ is supported in the winding region. The above lemma removes this restriction.

Lemma 3.5. *Suppose $\xi \in \underline{\mathrm{Spin}^c}(Y, K)$. For all sufficiently large m, there exists $\mathfrak{t} \in \mathrm{Spin}^c(Y_{m\mu+\lambda}(K))$, such that $\Xi(\mathfrak{t}) = \xi$.*

Proof. Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ be the underlying Spin^c structure of ξ. We can choose a Heegaard diagram for (Y, K) such that some $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ represents $\mathfrak{s}$, then $\xi = \underline{\mathfrak{s}}_{w,z}(\mathbf{x}) + n \cdot \mu$ for some $n \in \mathbb{Z}$. Now our desired result follows from the definition of Ξ. $\square$

The following proposition is a part of [13, Theorem 4.1].

Proposition 3.6. *Let $K \subset Y$ be a rationally null-homologous knot in a closed, oriented three-manifold, equipped with a framing λ. Let*

$$\widehat{A}_\xi(Y, K) = C_\xi\big\{\max\{i, j\} = 0\big\},$$

where $C_\xi = CFK^\infty(Y, K, \xi)$ as in [13]. Then, for all sufficiently large m and all $\mathfrak{t} \in \mathrm{Spin}^c(Y_{m\mu+\lambda}(K))$, there is an isomorphism

$$\Psi_{\mathfrak{t},m} \colon \widehat{CF}(Y_{m\mu+\lambda}(K), \mathfrak{t}) \to \widehat{A}_{\Xi(\mathfrak{t})}(Y, K).$$

3.2 Rational surgeries on null-homologous knots

Let K be a null-homologous knot in Y. As in [13, Section 7], $Y_{\frac{p}{q}}(K)$ can be realized by a Morse surgery with coefficient a on the knot $K' = K\#O_{q/r} \subset Y' = Y\#L(q,r)$, where $O_{q/r}$ is a U–knot in $L(q,r)$, $p = aq + r$. Let

$$\Xi' \colon \mathrm{Spin}^c(Y'_{a\mu'+\lambda'}) \to \underline{\mathrm{Spin}^c}(Y', K')$$

be the map defined in Definition 3.2.

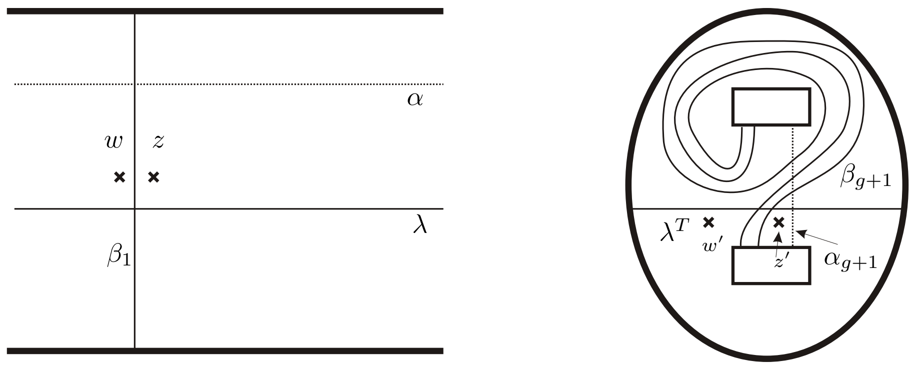

Figure 2: The left hand side is a piece of a Heegaard diagram for (Y,K). The right hand side is a genus 1 Heegaard diagram for $(L(q,r), O_{q/r})$. The boundary of the oval is capped off with a disk, and the boundaries of the two rectangles are glued together via a reflection. Here we choose $q=3, r=2$.

Construction 3.7. Let

$$(\Sigma, \boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}, \boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}, w, z)$$

be a doubly-pointed Heegaard diagram for (Y,K), such that β_1 is a meridian for K and the two base points are induced from a marked point on β_1. Suppose $\lambda \subset \Sigma$ represents a longitude of K.

Let

$$(T, \{\alpha_{g+1}\}, \{\beta_{g+1}\}, w', z')$$

be a genus 1 Heegaard diagram for $(L(q,r), O_{q/r})$. As in Figure 2, β_{g+1} intersects α_{g+1} exactly q times and intersects the boundary of each rectangle exactly r times. Suppose $\lambda^T \subset T$ represents a longitude of $O_{q/r}$.

We perform the connected sum of Σ and T by identifying the neighborhoods of z and w', hence we get a new genus $(g+1)$ surface Σ'. Then

$$(\Sigma', \boldsymbol{\alpha}' = \boldsymbol{\alpha} \cup \{\alpha_{g+1}\}, \boldsymbol{\beta}' = \boldsymbol{\beta} \cup \{\beta_{g+1}\}, w, z')$$

is a Heegaard diagram for (Y', K'). The longitude λ' of K' is a connected sum of λ and λ^T. $\square$

We define

$$\Pi_1\colon \underline{\mathrm{Spin}^c}(Y',K') \to \underline{\mathrm{Spin}^c}(Y,K)$$

as follows. Given $\xi' \in \underline{\mathrm{Spin}^c}(Y',K')$, suppose $\mathbf{x}' \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ represents the underlying Spin^c structure of ξ', then

$$\xi' = \underline{\mathfrak{s}}_{w,z'}(\mathbf{x}') + n \cdot \mu'$$

for some $n \in \mathbb{Z}$. Now let $\mathbf{x}$ be the projection of $\mathbf{x}'$ to $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, then

$$\Pi_1(\xi') = \underline{\mathfrak{s}}_{w,z}(\mathbf{x}) + n \cdot \mu.$$

The following proposition is obvious. (See also [13, Corollary 5.3].)

Proposition 3.8. *For any $\xi' \in \underline{\mathrm{Spin}^c}(Y',K')$, we have*

$$CFK^\infty(Y',K',\xi') \cong CFK^\infty(Y,K,\Pi_1(\xi'))$$

as $\mathbb{Z}\oplus\mathbb{Z}$–filtered chain complexes.

Lemma 3.9. *When m is sufficiently large, we have*

$$\pi = G_{Y,K} \circ \Pi_1 \circ \Xi'.$$

Here $G_{Y,K}\colon \underline{\mathrm{Spin}^c}(Y,K) \to \mathrm{Spin}^c(Y)$ is the map defined in [13, Section 2.2].

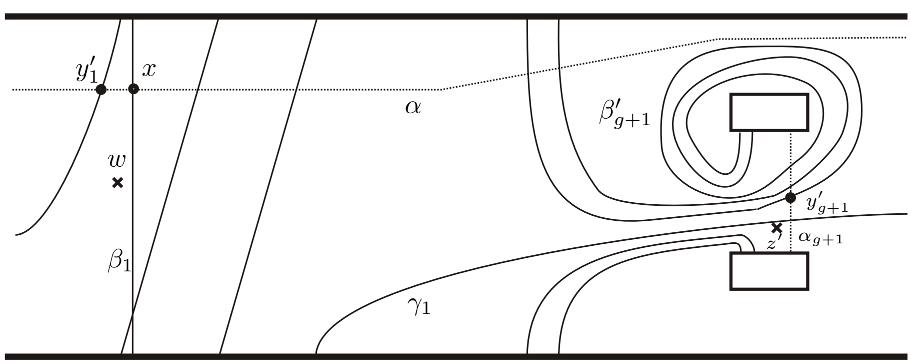

Figure 3: A Heegaard diagram for $Y'_{a\mu'+\lambda'}(K')$. Here we choose $a = 3$.

Proof. We follow the notation in Construction 3.7. Since λ' intersects β_1 exactly once, we can slide β_{g+1} over β_1 r times to eliminate the intersection points in $\beta_{g+1} \cap \lambda'$. The new curve is denoted β'_{g+1} as in Figure 3. Then

$$(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}'' = \boldsymbol{\beta} \cup \{\beta'_{g+1}\}, w, z')$$

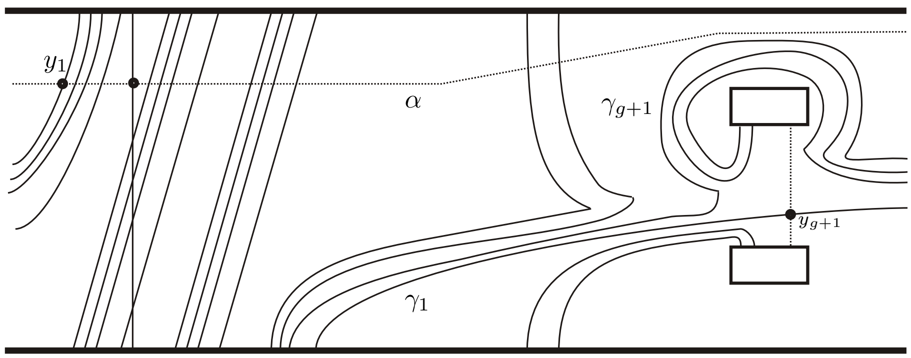

Figure 4: After q handleslides, we get a Heegaard diagram for $Y_{p/q}(K)$.

is also a Heegaard diagram for (Y', K'). Let $\gamma_1 = a\beta_1 + \lambda'$, then

$$(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\gamma}_1 = \{\gamma_1, \beta_2, \dots, \beta_g, \beta'_{g+1}\}, w)$$

is a Heegaard diagram for $Y'_{a\mu'+\lambda'}(K')$.

The curve α_{g+1} intersects γ_1 exactly once. We can slide β'_{g+1} over γ_1 q times to eliminate its q intersection points with α_{g+1}. The new curve is denoted γ_{g+1} as in Figure 4. Now

$$(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\gamma}_2 = \{\gamma_1, \beta_2, \dots, \beta_g, \gamma_{g+1}\}, w)$$

is a Heegaard diagram for $Y'_{a\mu'+\lambda'}(K') = Y_{p/q}(K)$. Moreover, we may slide other α–curves over α_{g+1} to eliminate their intersection points with γ_1. A destabilization will remove α_{g+1} and γ_1. Now we get a diagram

$$(\Sigma^*, \boldsymbol{\alpha}^*, \boldsymbol{\gamma}^*, w)$$

which is isomorphic to

$$(\Sigma, \boldsymbol{\alpha}, \{\beta_2, \dots, \beta_g, \gamma^*_{g+1}\}, w),$$

where γ^*_{g+1} is the image of γ_{g+1} under the destabilization.

We want to show that γ^*_{g+1} is isotopic to $p\mu + q\lambda$, the curve obtained by doing cut-and-pastes to p parallel copies of μ and q parallel copies of λ. In fact, γ^*_{g+1} is supported in a small neighborhood of $\mu \cup \lambda$, so it must be isotopic to $p'\mu + q'\lambda$ for some p', q'. It is easy to compute the intersection numbers of γ_{g+1} with λ and $\mu = \beta_1$, which are $p = aq + r$ and q. The intersection numbers of γ^*_{g+1} with μ and λ remains the same, so $\gamma^*_{g+1} = p\mu + q\lambda$.

Suppose $\mathfrak{t} \in \mathrm{Spin}^c(Y_{p/q}(K))$. We want to prove

$$\pi(\mathfrak{t}) = G_{Y,K} \circ \Pi_1 \circ \Xi'(\mathfrak{t}). \tag{4}$$

We first consider the right hand side of (4). Let $\mathbf{y}'$ be a point in $\mathbb{T}_{\alpha'}\cap\mathbb{T}_{\gamma_1}$ which is supported in the winding region and represents $\mathfrak{t}$ (Figure 3). Suppose the γ_1–coordinate of $\mathbf{y}'$ is y_1' and the β'_{g+1}–coordinate is y'_{g+1}.

Let $\mathbf{x}'\in\mathbb{T}_{\alpha'}\cap\mathbb{T}_{\beta''}$ be the nearest point to $\mathbf{y}'$, then (2) implies that

$$\Xi'(\mathfrak{t})=\underline{\mathfrak{s}}_{w,z'}(\mathbf{x}')+n\cdot\mu'$$

for some $n\in\mathbb{Z}$. Let $\mathbf{x}$ be the projection of $\mathbf{x}'$ to $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$, then

$$\Pi_1\circ\Xi'(\mathfrak{t})=\underline{\mathfrak{s}}_{w,z}(\mathbf{x})+n\cdot\mu.$$

Hence

$$G_{Y,K}\circ\Pi_1\circ\Xi'(\mathfrak{t})=\mathfrak{s}_w(\mathbf{x}).$$

Now we consider the left hand side of (4). As in Figure 4, we get another Heegaard diagram for $Y_{p/q}(K)$ by q handle slides. In this diagram, we can find a point $\mathbf{y}\in\mathbb{T}_{\alpha'}\cap\mathbb{T}_{\gamma_2}$ which represents $\mathfrak{t}$ as $\mathbf{y}'$ does. In fact, since α_{g+1} intersects γ_1 exactly once and is disjoint from other γ–curves, $\mathbf{y}$ must contain the intersection point of α_{g+1} and γ_1, denoted y_{g+1}. The γ_1–coordinate of $\mathbf{y}$, called y_1, is determined by y_1' and y'_{g+1}: it is one of the q intersection points on γ_{g+1} near y_1', and the choice among these q points is specified by the position of y'_{g+1}. Other coordinates of $\mathbf{y}$ are the same as $\mathbf{y}'$.

After handleslides and one destabilization, we get a point $\mathbf{y}^*\in\mathbb{T}_{\alpha^*}\cap\mathbb{T}_{\gamma^*}$ whose coordinates are the same as $\mathbf{x}$ except that its γ_1–coordinate is y_1. So its nearest point in $\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ is $\mathbf{x}$, hence $\mathbf{x}$ represents $\pi(\mathfrak{t})$. This proves (4). □

Lemma 3.10. *Let $H(\widehat{A}_\xi(Y,K))$ be the homology of the chain complex $\widehat{A}_\xi(Y,K)$. For a fixed ξ, when $|n|\gg 0$,*

$$H(\widehat{A}_{\xi+n\cdot\mu}(Y,K))\cong\widehat{HF}(Y,G_{Y,K}(\xi)).$$

Proof. By the definitions

$$\begin{aligned}\widehat{A}_{\xi+n\cdot\mu}(Y,K)&=C_{\xi+n\cdot\mu}\{\max\{i,j\}=0\}\\&=C_\xi\{\max\{i,j-n\}=0\}.\end{aligned}$$

By the adjunction inequality, $H(C_\xi\{i,j\})=0$ when $|i-j|\gg 0$. So

$$H(C_\xi\{\max\{i,j-n\}=0\})\cong H(C_\xi\{i=0\})$$

when $n\gg 0$. The latter group is isomorphic to $\widehat{HF}(Y,G_{Y,K}(\xi))$ by [13, Proposition 3.2].

When $n\ll 0$, we have

$$H(C_\xi\{\max\{i,j-n\}=0\})\cong H(C_\xi\{j=n\})\cong H(C_\xi\{j=0\}),$$

which is isomorphic to $\widehat{HF}(Y,G_{Y,-K}(\xi))$ by [13, Proposition 3.2]. Now by [13, Equation (4)] and the fact that K is null-homologous, we have $G_{Y,K}(\xi)=G_{Y,-K}(\xi)$. □

Lemma 3.11. *Suppose $\widehat{HF}(Y,\mathfrak{s})=0$, then $H(\widehat{A}_{\xi'}(Y',K'))\neq 0$ for only finitely many $\xi'\in(G_{Y,K}\circ\Pi_1)^{-1}(\mathfrak{s})$.*

Proof. For each $\xi\in\underline{\mathrm{Spin}}^c(Y,K)$, there are exactly q relative Spin^c structures in $\Pi_1^{-1}(\xi)$. Moreover, by Proposition 3.8, if $\xi'\in\Pi_1^{-1}(\xi)$, then

$$\widehat{A}_{\xi'}(Y',K')\cong\widehat{A}_{\xi}(Y,K).$$

Hence we only need to show that $H(\widehat{A}_\xi(Y,K))\neq 0$ for only finitely many $\xi\in G_{Y,K}^{-1}(\mathfrak{s})$.

Pick any $\xi\in G_{Y,K}^{-1}(\mathfrak{s})$, then

$$G_{Y,K}^{-1}(\mathfrak{s})=\{\xi+i\cdot\mu|\ i\in\mathbb{Z}\}.$$

By Lemma 3.10, $H(\widehat{A}_{\xi+i\cdot\mu}(Y,K))$ is isomorphic to $\widehat{HF}(Y,\mathfrak{s})$ when $|i|$ is large, hence is 0. This finishes the proof. □

Proposition 3.12. *When m is sufficiently large,*

$$\begin{aligned}\widehat{HF}(Y'_{m\mu'+\lambda'}(K'),\pi^{-1}(\mathfrak{s})) &\cong \bigoplus_{\{\xi'|\ G_{Y,K}\circ\Pi_1(\xi')=\mathfrak{s}\}} H(\widehat{A}_{\xi'}(Y',K'))\\ &\cong \bigoplus^{q}\bigoplus_{\{\xi|\ G_{Y,K}(\xi)=\mathfrak{s}\}} H(\widehat{A}_{\xi}(Y,K)).\end{aligned}$$

Proof. By Proposition 3.6, when m is sufficiently large

$$\widehat{HF}(Y'_{m\mu'+\lambda'}(K'),\pi^{-1}(\mathfrak{s}))\cong\bigoplus_{\mathfrak{t}\in\pi^{-1}(\mathfrak{s})}H(\widehat{A}_{\Xi'(\mathfrak{t})}(Y',K')).$$

By Lemma 3.9,

$$\Xi'(\pi^{-1}(\mathfrak{s}))=\Xi'\left(\Xi'^{-1}\circ(G_{Y,K}\circ\Pi_1)^{-1}(\mathfrak{s})\right)\subset(G_{Y,K}\circ\Pi_1)^{-1}(\mathfrak{s}).$$

Consider the map

$$\Xi'_{\mathfrak{s}}\colon\ \pi^{-1}(\mathfrak{s})\to(G_{Y,K}\circ\Pi_1)^{-1}(\mathfrak{s}).$$

By [8, Lemma 2.4], $\Xi'_{\mathfrak{s}}$ is injective. Moreover, by Lemmas 3.5 and 3.11, when m is sufficiently large, the range of $\Xi'_{\mathfrak{s}}$ contains all $\xi'\in(G_{Y,K}\circ\Pi_1)^{-1}(\mathfrak{s})$ satisfying $H(\widehat{A}_{\xi'}(Y',K'))\neq 0$. This proves the first equality.

In order to prove the second equality, we note that for each $\xi\in\underline{\mathrm{Spin}}^c(Y,K)$, there are exactly q relative Spin^c structures in $\Pi_1^{-1}(\xi)$. Moreover, by Proposition 3.8, if $\xi'\in\Pi_1^{-1}(\xi)$, then

$$\widehat{A}_{\xi'}(Y',K')\cong\widehat{A}_{\xi}(Y,K).$$

So the second equality easily follows. □

Proof of Theorem 3.1. Let

$$C = \operatorname{rank} \bigoplus_{\{\xi \mid G_{Y,K}(\xi)=\mathfrak{s}\}} H(\widehat{A}_\xi(Y,K)).$$

By Proposition 3.12,

$$\operatorname{rank} \widehat{HF}(Y_{p/q}, \pi^{-1}(\mathfrak{s})) = qC$$

when p is sufficiently large.

Since $\widehat{HF}(Y,\mathfrak{s}) = 0$, we have $\widehat{HF}(Y',\mathfrak{s}') = 0$ for any $\mathfrak{s}'$ that extends $\mathfrak{s}$. By [10, Theorem 9.12], we have the long exact sequence

$$\begin{tikzcd}
\widehat{HF}(Y', P_1^{-1}(\mathfrak{s})) \arrow[r] & \widehat{HF}(Y'_{m\mu'+\lambda'}(K'), \pi_m^{-1}(\mathfrak{s})) \arrow[dl] \\
\widehat{HF}(Y'_{(m+1)\mu'+\lambda'}(K'), \pi_{m+1}^{-1}(\mathfrak{s})) \arrow[u] &
\end{tikzcd},$$

where

$$P_1\colon \operatorname{Spin}^c(Y') \to \operatorname{Spin}^c(Y),$$

$$\pi_m\colon \operatorname{Spin}^c(Y'_{m\mu'+\lambda'}(K')) \to \operatorname{Spin}^c(Y)$$

are the natural projection maps. Since $\widehat{HF}(Y', P_1^{-1}(\mathfrak{s})) = 0$, we have

$$\widehat{HF}(Y'_{a\mu'+\lambda'}(K'), \pi_a^{-1}(\mathfrak{s})) \cong \widehat{HF}(Y'_{m\mu'+\lambda'}(K'), \pi_m^{-1}(\mathfrak{s}))$$

for m sufficiently large. Hence its rank is always qC. □

4 Cosmetic surgeries

Proof of Theorem 1.3. Assume there are two rational numbers $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ satisfying that there is a homeomorphism

$$f\colon Y_{\frac{p_1}{q_1}} \to \pm Y_{\frac{p_2}{q_2}},$$

then $|p_1| = |p_2|$ for homological reasons. If $\frac{p_1}{q_1} \neq \pm\frac{p_2}{q_2}$, then we can assume

$$0 < q_1 < q_2.$$

Without loss of generality, we may assume $Y - K$ is irreducible. By (1) and the adjunction inequality, we conclude that $\widehat{HF}(Y, h, \frac{1}{2}x_{Y-K}(h)) = 0$. It then follows from Theorem 3.1 that there is a constant C_h, such that

$$\operatorname{rank} \widehat{HF}(Y_{p/q}(K), h, \frac{1}{2}x_{Y-K}(h)) = qC_h.$$

Since (1) holds, [1, Corollary 2.4] implies that

$$x_{Y-K}(h) = x_{Y_{p/q}(K)}(h)$$

for any nonzero $h \in H_2(Y)$ and $\dfrac{p}{q} \in \mathbb{Q}$. Theorem 2.3 then implies that

$$\operatorname{rank} \widehat{HF}(Y_{p/q}(K)|h) = qC_h \neq 0.$$

Since K is null-homologous, the inclusion maps $Y - K \to Y_r$ induce isomorphisms on H_2 for each $r \in \mathbb{Q} \cup \{\infty\} \backslash \{0\}$. Hence we can identify $H_2(Y_r(K))$ with $H_2(Y)$. Now $f_*\colon\ H_2(Y_{\frac{p_1}{q_1}}) \to H_2(Y_{\frac{p_2}{q_2}})$ can be regarded as a map

$$f_*\colon\ H_2(Y) \to H_2(Y).$$

Fix a nonzero $h \in H_2(Y)$, we have

$$\operatorname{rank} \widehat{HF}(Y_{\frac{p_1}{q_1}}|f_*^n(h)) = \frac{q_1}{q_2}\operatorname{rank} \widehat{HF}(Y_{\frac{p_2}{q_2}}|f_*^n(h)) \neq 0$$

for any $n \in \mathbb{Z}$. Moreover, since $f\colon\ Y_{\frac{p_1}{q_1}} \to \pm Y_{\frac{p_2}{q_2}}$ is a homeomorphism, we have

$$\operatorname{rank} \widehat{HF}(Y_{\frac{p_1}{q_1}}|f_*^{n-1}(h)) = \operatorname{rank} \widehat{HF}(Y_{\frac{p_2}{q_2}}|f_*^n(h)).$$

Thus we get

$$\operatorname{rank} \widehat{HF}(Y_{\frac{p_1}{q_1}}|f_*^n(h)) = \left(\frac{q_1}{q_2}\right)^n \operatorname{rank} \widehat{HF}(Y_{\frac{p_1}{q_1}}|h) \neq 0.$$

So $0 < \operatorname{rank}\widehat{HF}(Y_{\frac{p_1}{q_1}}|h) < 1$ when n is sufficiently large, which is impossible. $\square$

Proof of Theorem 1.4. Since $x_Y \equiv 0$, the adjunction inequality implies that $\widehat{HF}(Y, h, \frac{1}{2}x_{Y-K}(h)) = 0$ for any $h \in H_2(Y)$ satisfying $x_{Y-K}(h) \neq 0$. Using Theorems 3.1, 2.3 and [1, Corollary 2.4], we have

$$\operatorname{rank} \widehat{HF}(Y_{p/q}(K)|h) = qC_h$$

for some nonzero constant C_h. Now the argument is the same as in the proof of Theorem 1.3. $\square$

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